The wave field outside the stamp is representable as

$$u(\rho, t) = \alpha \mathrm{Re}\, e^{-i\omega t} \sum_{j=1}^{p} A_j H_0^{(1)}(p_j \rho) \qquad (2.7)$$

Let us consider the case $\nu = 0.2, a = 1, x_2^2 = 11$. In this case the function $K(u)$ has the poles $p_1 = 0.4984, p_2 = 2.004, p_3 = 3.393$ and the zeros $z_1 = 2.031, z_2 = 1.063$ on the real semi-axis. For the approximation $M^2 = 110, B = 15, z_3 = 2.397\,i$. The error will not exceed 6% for approximation by a fourth degree polynomial.

Formulas (2.6) and (2.7) permit computation of the stresses under the stamp in a domain not adjoining its edge. Upon approaching the edge of the stamp, the stresses grow as $(r-a)^{-1/2}$ and this singularity can easily be isolated by using the method [5], for example.

Waves on the layer surface in the far zone are computed by means of (2.7). To compute the field in the nearest zone it is necessary to use (2.5); this formula can describe the wave field in a zone arbitrarily near to the stamp because the parameter B increases.

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OPTIMIZATION OF VIBRATION FREQUENCIES OF AN ELASTIC PLATE IN AN IDEAL FLUID

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The problem of optimizing the frequencies of an elastic plate vibrating in an ideal fluid is investigated. A formulation of the appropriate hydroelasticity problem is presented. The "external" hydrodynamic problem is solved by methods of complex variable function theory and the forces exerted by the fluid on the plate are determined. An integro-differential equation describing one-dimensi-

onal plate vibrations in a fluid is derived. A formulation and investigation of the optimization problem are given. A numerical argorithm to determine the optimal modes is elucidated and results of computations on an electronic computer are presented.

Determination of the optimal modes of elastic bodies whose fundamental natural vibrations frequency is a maximum (for a given volume of weight) is of interest in connection with some problems of designing structures working under dynamic conditions, and in particular, in flutter problems. The first investigations in this area were carried out in [1 — 3] in the example of optimizing the natural frequencies of a string. The corresponding optimization problems for elastic beams were solved in [4 — 9] for different kinds of vibrations (transverse, longitudinal and torsional) and boundary conditions. Plate optimization problems were investigated in [10 — 12]. The dual problem of weight minimization for a given fundamental tone frequency was considerd in some of the papers listed.

1. The problem of plate vibrations in an ideal fluid. Let us consider the plane hydroelasticity problem concerning small vibrations of a thin elastic plate in an infinite ideal fluid volume. As is known, the investigation of the cylindrical strain of a long rectangular plate, clamped along the long edges, and of the plane fluid vibrations which originate, reduces to the problem mentioned. The xy-plane in which the plate and fluid vibrations occur is perpendicular to the long edges of the plate. The points A $(x = -l,\ y = 0)$ and B $(x = l,\ y = 0)$ on the xy-plane correspond to clamped edges. The plate width is $2l$. It is assumed that the plate thickness is variable in the x-coordinate, i.e. $h = h(x)$, and does not vary in the z-direction (the axis is parallel to the fixed edges of the plate). Letting t denote the time, and $u = u(x, t)$ and $Q = Q(x, t)$ the plate deflection and fluid reaction functions, respectively, let us write the equation for cylindrical plate vibrations

$$\rho_1 h \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(D \frac{\partial^2 u}{\partial x^2}\right) = Q, \quad D = \frac{Eh^3}{12(1-\nu^2)} \tag{1.1}$$

Here D is the bending stiffness of the plate, E is Young's modulus, ν is the Poisson's ratio, ρ_1 is the specific density of the plate material. For definiteness, let us take the following boundary conditions at the points A and B, correspondint to hinge-support of the plate edges

$$u = h^3 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

The fluid motion is assumed irrotational, with the velocity potential $\varphi = \varphi(x, y, t)$ satisfying the Laplace equation and the linearized boundary conditions

$$\partial^2\varphi / \partial x^2 + \partial^2\varphi / \partial y^2 = 0 \tag{1.3}$$

$$(\partial\varphi / \partial y)^\pm = \partial u / \partial t, \quad -l \leqslant x \leqslant l, \quad y = 0 \tag{1.4}$$

Here and henceforth, the plus and minus superscripts will denote values of the appropriate quantities on the upper and lower edges of the slit $-l \leqslant x \leqslant l$, $y = 0$. The boundary conditions (1.4) are obtained by moving the boundary conditions of non-penetration of the fluid through the plate surface on the x-axis. Assumptions about the smallness of the diflections u and the thickness h, the smoothness of the function $h(x)$, and also the inseparability of the fluid and plate motions are used here.

The pressure distribution in the fluid $p = p(x, y, t)$ is calculated by using the Cauchy-Lagrange integral

$$p = p_\infty - \rho_2\left[\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2\right] \approx p_\infty - \rho_2\frac{\partial\varphi}{\partial t} \tag{1.5}$$

where ρ_2 is the fluid density, p_∞ is the fluid pressure at infinity. Neglecting of terms $\rho_2(\nabla\varphi)^2$ is based on the assumption of smallness of vibrations and the use of linear theory. The relationship (1.5) permits determining p if the potential φ has been found. The fluid reaction Q in the right side of the vibrations equation (1.1) is determined by the linearized formula

$$Q = p^- - p^+ \tag{1,6}$$

where $p^+ = (p(x, 0, t))^+$ and $p^- = (p(x, 0, t))^-$ denote the pressure distributions for the upper and lower edges of the slit $-l \leqslant x \leqslant l,\ y = 0$, respectively. Determining Q from (1.5), (1.6) and substituting the expression found into (1.1), we convert the plate vibrations equations into

$$\rho_1 h\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 u}{\partial x^2}\right) = \rho_2\left(\frac{\partial\varphi^+}{\partial t} - \frac{\partial\varphi^-}{\partial t}\right) \tag{1,7}$$

We now have the closed boundary value problem of hydroelasticity. The hydrodynamic problem (1.3), (1.4) to determine the potential φ and the problem (1.2), (1.7) on plate vibrations are connected, since the velocity distributions of the plate vibrations enter into the boundary conditions (1.4) and the derivatives of the potential φ are in the right side of (1.7).

Let us seek the solution of the problem (1.2) — (1.4), (1.7) as

$$u = e^{i\omega t}\,U(x), \qquad \varphi = i\omega e^{i\omega t}\,\Phi(x, y) \tag{1.8}$$

For convenience in the analysis and solution of the problem, let us pass to dimensionless variables and the notation

$$x' = x/l,\ y' = y/l,\ U' = U/l,\ \Phi' = \Phi/l^2,\ h' = lh/S \tag{1.9}$$
$$\Omega^2 = 12\rho_1 l^6(1 - \nu^2)\omega^2/S^2E, \qquad \alpha = \rho_2 l^2/\rho_1 S$$

where S is the cross-sectional area of the plate. The primes on the dimensionless variables are henceforth omitted.

Substitution of (1.8), (1.9) into (1.2) — (1.4), (1.7) results in the following relationships to determine the amplitude functions $U(x)$ and $\Phi(x, y)$:

$$\frac{d^2}{dx^2}\left(h^3\frac{d^2U}{dx^2}\right) - \Omega^2[hU - \alpha(\Phi^+ - \Phi^-)] = 0 \tag{1.10}$$

$$U|_{x=\pm1} = \left(h^3\frac{d^2U}{dx^2}\right)_{x=\pm1} = 0 \tag{1.11}$$

$$\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 = 0 \tag{1.12}$$

$$(\partial\Phi/\partial y)^\pm = U, \quad -1 \leqslant x \leqslant 1, \quad y = 0 \tag{1.13}$$

Therefore, by separating the space and time variables, the initial problem (1.2) — (1.4), (1.7) is reduced to the boundary value problem (1.10), (1.11) for eigenvalues of the differential equation (1.10) under the boundary conditions (1.11) and its related boundary value problem (1.12), (1.13) for the Laplace equation in the exterior of the slit $-1 \leqslant x \leqslant 1,\ y = 0$.

2. Solution of the hydrodynamic problem. Let us determine the (potential) function $\Phi(x, y)$, which is a solution of the boundary value problem (1.12), (1.13). To this end, let us consider the auxiliary function $W = \Psi + i\Phi$, which is analytic in a plane with the slit $-1 \leqslant x \leqslant 1,\ y = 0$.

From the Cauchy-Riemann conditions and the boundary conditions (1.13) we will have

$$\partial\Psi / \partial x = \partial\Phi / \partial y = U$$

from which

$$\Psi = \int_{-1}^{x} U(\xi)\, d\xi + C = f(x) + C \tag{2.1}$$

where C is a constant of integration. The problem of determining the potential Φ reduces to seeking the imaginary part of the analytic function W whose real part in the segment $[-1, 1]$ is

$$\operatorname{Re} W = \Psi = f(x) + C$$

Using the results in [13],[14], let us write down the solution of this problem ($z = x + iy$)

$$W = \frac{1}{2\pi i}\left(\frac{z-1}{z+1}\right)^{1/2} \int_{-1}^{1} \left(\frac{t+1}{t-1}\right)^{1/2} \frac{f(t) + C}{t - z}\, dt \tag{2.2}$$

$$\frac{1}{2\pi i} \int_{-1}^{1} \frac{f(t) + C}{\sqrt{t^2 - 1}}\, dt = 0 \tag{2.3}$$

The relationship (2.3) determines the constant C.

Taking into account that

$$-\frac{1}{\pi i} \int_{-1}^{1} \frac{dt}{\sqrt{t^2 - 1}} = 1$$

we obtain from (2.3)

$$C = \frac{1}{\pi i} \int_{-1}^{1} \frac{f(\xi)\, d\xi}{\sqrt{\xi^2 - 1}} \tag{2.4}$$

Furthermore, using the expression (2.4) for C and the formula

$$\frac{1}{2\pi i} \int_{-1}^{1} \left(\frac{\xi+1}{\xi-1}\right)^{1/2} \frac{d\xi}{\xi - z} = \frac{1}{2}\left(\frac{z+1}{z-1}\right)^{1/2} - \frac{1}{2}$$

we perform the following manipulations in the representation (2.2) for W:

$$W = \frac{1}{2\pi i}\left(\frac{z-1}{z+1}\right)^{1/2} \int_{-1}^{1} \left(\frac{\xi+1}{\xi-1}\right)^{1/2} \frac{f(\xi)\, d\xi}{\xi - z} + \frac{C}{2}\left[1 - \left(\frac{z-1}{z+1}\right)^{1/2}\right] = \tag{2.5}$$

$$\frac{1}{2\pi i}\left(\frac{z-1}{z+1}\right)^{1/2} \int_{-1}^{1} f(\xi) \left(\frac{\xi+1}{\xi-1}\right)^{1/2} \left(\frac{1}{\xi - z} - \frac{1}{\xi + 1}\right) d\xi + \frac{C}{2} =$$

$$\frac{\sqrt{z^2 - 1}}{2\pi i} \int_{-1}^{1} \frac{f(\xi)\, d\xi}{(\xi - z)\sqrt{\xi^2 - 1}} + \frac{C}{2}$$

Let us evaluate the quantity Φ^+ by passing to the limit in (2.5) as $z = x + iy \to x + i0$ $(0 < y \to +0)$ and extracting the imaginary part in the expression obtained

$$\Phi^+ = \lim_{y \to +0} [\operatorname{Im} W(x + iy)] = \mathrm{vp} - \frac{\sqrt{1 - x^2}}{2\pi} \int_{-1}^{1} \frac{f(\xi)\, d\xi}{(\xi - x)\sqrt{1 - \xi^2}} \tag{2.6}$$

The integral in (2.6) is understood in the sense of the Cauchy principal value. We have the following for the potential difference on the upper (plus) and lower (minus) edges of the slit

$$\Phi^+ - \Phi^- = 2\Phi^+ = \mathrm{vp} - \frac{\sqrt{1-x^2}}{\pi} \int_{-1}^{1} \frac{f(\xi)\, d\xi}{(\xi - x)\sqrt{1-\xi^2}} \tag{2.7}$$

Let us convert the integral (2.7). By definition of an integral in the principal value sense, we have

$$2\Phi^+ = \mathrm{vp}\frac{1}{\pi} \int_{-1}^{1} \left(\frac{1-x^2}{1-t^2}\right)^{1/2} \frac{f(t)\,dt}{t-x} =$$

$$\lim_{\varepsilon\to 0} \frac{1}{\pi}\left[\int_{-1}^{x-\varepsilon} \left(\frac{1-x^2}{1-t^2}\right)^{1/2} \frac{f(t)\,dt}{t-x} + \int_{x+\varepsilon}^{1} \left(\frac{1-x^2}{1-t^2}\right)^{1/2} \frac{f(t)\,dt}{t-x}\right]$$

Furthermore, integrating by parts and using (2.1) for $f(t)$, we obtain

$$2\Phi^+ = \lim_{\varepsilon\to 0}\Big[K(x-\varepsilon, x) \int_{-1}^{x-\varepsilon} U(t)\,dt - K(x+\varepsilon, x) \int_{-1}^{x+\varepsilon} U(t)\,dt - \tag{2.8}$$

$$\int_{-1}^{x-\varepsilon} K(t, x)\,U(t)\,dt - \int_{x+\varepsilon}^{1} K(t, x)\,U(t)\,dt\Big]$$

$$K(t, x) = \frac{1}{\pi}\ln\left|\frac{1+r}{1-r}\right|, \quad r = \left[\frac{(1-x)(1+t)}{(1-t)(1+x)}\right]^{1/2}$$

Let us note that all the expressions in the right side of (2.8) (written in the square brackets) are finite, and therefore, the integration by parts used above is possible. As $\varepsilon \to 0$ the sum of the first two terms is (2.8) vanishes and the last two integrals converge. Therefore, the desired dependence of the value of the potential jump $\Phi^+ - \Phi^- = 2\Phi^+$ on the plate deflection distribution is

$$2\Phi^+ = -\int_{-1}^{1} K(t, x)\,U(t)\,dt \tag{2.9}$$

3. Formulation of the optimization problem. Using the results of Sect. 2, let us write down the beam vibrations equation. Substituting the expression (2.9) found for the difference in the hydrodynamic potentials into (1.10), we have

$$LU = \frac{d^2}{dx^2}\left(h^3\frac{d^2U}{dx^2}\right) - \Omega^2\left(hU + \alpha\int_{-1}^{1} K(t, x)\,U(t)\,dt\right) = 0 \tag{3.1}$$

Let us consider the homogeneous boundary value problem for the eigenvalues of the integro-differential equation (3.1) under the boundary conditions (1.11). The eigenfunctions $U(x)$ and the corresponding eigennumbers Ω (frequencies) are determined from the solution of the problem (3.1),(1.11) for a given thickness distribution $h = h(x)$. From the equation $(U, LU) = 0$ we determine the Rayleigh quotient

$$\Omega^2 = J(h, U) = I(h, U)\Big[\int_{-1}^{1} hU^2\,dx + \tag{3.2}$$

$$\alpha\int_{-1}^{1}\int_{-1}^{1} K(x, t)\,U(x)\,U(t)\,dx\,dt\Big]^{-1}$$

$$I(h, U) = \int_{-1}^{1} h^3 \left(\frac{d^2U}{dx^2}\right)^2 dx$$

The minimal eigenvalue (fundamental frequency) of the self-adjoint problem (3.1), (1.11) (for a given function $h = h(x)$) equals

$$\Omega^2 = \min_U J(h, U) \tag{3.3}$$

The relationship (3.3) is the known Rayleigh variational principle (see [15], for example). The minimum of J in U is calculated in the set of all continuous twice-differentiable functions $U(x)$ satisfying the boundary conditions

$$U(1) = U(-1) = 0 \tag{3.4}$$

It is not required to satisfy the boundary conditions $h^3(d^2U / dx^2) = 0$ imposed at $x = \pm 1$ in advance, since these conditions are "natural" for the functional J in(3.2) and the desired extremum of this functional, found under the conditions (3.4), will automatically satisfy the conditions $h^3(d^2U / dx^2) = 0$ at the points $x = -1$ and $x = 1$. The function $U(x)$ realizing this minimum will correspond to the minimal eigenvalue. The Euler equation for the function $U(x)$ in the vibrational problem (3.2) — (3.4) agrees with the equation (3.1).

The optimization problem solved below consists of seeking a function $h(x)$ satisfying the isoperimetric condition of constant cross-sectional area of the plate

$$\int_{-1}^{1} h\,dx = 1 \tag{3.5}$$

and maximizing the minimal eigenvalue, i.e.

$$\Omega^2 = \max_h \min_U J(h, U) \tag{3.6}$$

The formulated optimization problem (3.2) — (3.6) is one-parametric with the parameter $\alpha = \rho_2 l^2 / \rho_1 S$. For $\alpha = 0$ (the case of no fluid), we arrive at the known optimization problem for the fundamental frequency considered first in [4]. The difference existing in comparison to [4] in the case $\alpha = 0$ is due to the difference in the dependences $D = D(h)$ taken (the optimization problem for beams of circular cross section was considered in [4], for which $D \sim h^2$, where h is a variable radius of the section).

Let us obtain the necessary optimality condition in the problem (3.2 — 3.6). To do this, writing down the Euler equation in h for the functional (3.2) with the isoperimetric condition (3.5), we have

$$3h^2(d^2U / dx^2)^2 - \Omega^2 U^2 = c^2$$

where c^2 is the Lagrange multiplier corresponding to the isoperimetric condition (3.5).

4. Method of the numerical solution. Let us derive the fundamental relationships used in the numerical solution of the optimization problem. Let the function $h = h_0(x)$ in (3.2) satisfy the isoperimetric condition (3.5), and let $U = U_0(x)$ be the solution of the variational problem (3.2) — (3.4) for given $h = h_0(x)$. In addition to $h_0(x)$, let us consider the function $h_1(x)$ defined by the formula

$$h_1 = h_0 + \delta h \tag{4.1}$$

where δh denotes a small variation in the function h. In order for the function $h_1(x)$ to satisfy the isoperimetric condition (3.5), let us impose the following constraint on the function δh:

$$\int_{-1}^{1} \delta h\, dx = 0 \tag{4.2}$$

Let us calculate the variation of the functional (3.2) due to the variation in the thickness distribution (4.1).

Using the Euler equation (3.1) for the function U_0 and performing elementary manipulations, associated with evaluation of the first variation of the functional, we obtain

$$\delta\Omega^2 = \int_{-1}^{1} \Lambda \delta h\, dx, \quad \Lambda \equiv \left[3h^2 \left(\frac{d^2U}{dx^2}\right)^2 - \Omega^2 U^2\right] \Big/ I(h, U) \tag{4.3}$$

If the variation δh is given as ($\sigma > 0$ is a small positive number)

$$\delta h = \sigma\left(\Lambda - \frac{1}{2}\int_{-1}^{1} \Lambda\, dx\right) \tag{4.4}$$

then it is easy to see by direct substitution that condition (4.2) will be satisfied and the linear part of the increment in the functional, i.e. its first variation, is non-negative

$$\delta\Omega^2 \geqslant 0 \tag{4.5}$$

Therefore, selecting a sufficiently small positive number σ and evaluating the new thickness distribution by means of (4.1), (4.3), (4.4), the growth of the functional being optimized can be achieved. Condition (4.5) will hence be satisfied for any positive σ.

The relationships (4.1), (4.3), (4.4) permit construction of the following algorithm to seek the optimal thickness distribution $h = h(x)$. An initial approximation h_0 is given for the function $h(x)$ which satisfies the isoperimetric condition (3.5). For a given $h = h_0(x)$ the variational problem (3.2) — (3.4) is solved and the optimal distribution $U = U_0(x)$ corresponding to the thickness distribution $h = h_0(x)$ is found (the description of the method of solving the variational problem (3.2) — (3.4) is presented below). Furthermore, using the $U_0(x)$ distribution found, the variation δh is calculated by means of (4.3), (4.4) and a new approximation $h_1 = h_0 + \delta h$ is found for the thickness distribution. By solving the problem (3.2) — (3.4), a new approximation $U_1(x)$ for the function $U(x)$ is sought by means of the distribution h_1 found, etc. The calculation process described is repeated cyclically and is terminated upon compliance with the condition ($\varepsilon > 0$ is a sufficiently small number)

$$|J(h_i U_i) - J(h_{i-1}, U_{i-1})| < \varepsilon \tag{4.6}$$

Therefore, the optimization algorithm described reduces to the successive solution of the variational problem (3.2) — (3.4), the calculation of new approximations for the function h by means of (4.1) — (4.4) and verification of (4.6). Let us note that in carrying out the computations according to this algorithm a check on the residuals in complying with the necessary extremal conditions is also realized.

The method of local variations ([16]) is used to solve the variational problems (3.2) — (3.4). Symmetry of the problem (3.2) — (3.4) relative to the point $x = 0$ was used and the solution was sought on the segment $-1 \leqslant x \leqslant 0$. The segment $[-1, 0]$ is divided into n equal cells. Centers of the cells with the coordinates $x_i = -(n + 0{,}5 - i)/n$, $i = 0, 1, \ldots, n + 1$ were selected as nodes of the mesh. The introduction of fictitious points x_0 and x_{n+1} which do not belong to the segment $[-1, 0]$ is for convenience in the finite difference approximation and standardization of the calculations. The central differences

$$\left(\frac{d^2U}{dx^2}\right)_{x=x_i} = (U_{i+1} + U_{i-1} - 2U_i)/(\Delta x)^2, \quad U_i = U(x_i)$$

were used to approximate the derivatives.

The integrals in (3.2) for J were replaced by the following quadratures

$$\int_{-1}^{0} h^3 \left(\frac{d^2U}{dx^2}\right)^2 dx \approx \sum_{1}^{n} h_i^3 \left[\frac{U_{i+1} + U_{i-1} - 2U_i}{(\Delta x)^2}\right]^2 \Delta x$$

$$\int_{-1}^{0} hU^2 dx \approx \sum_{1}^{n} h_i U_i^2 \Delta x$$

$$\int_{-1}^{0}\int_{-1}^{0} K(x, t)\, U(x)\, U(t)\, dx dt \approx \sum_{i, j=1}^{n} K_{ij} U_i U_j (\Delta x)^2$$

The step in the variation of the function was changed a half division between 10^{-1} and 10^{-6} in solving the problem by the method of local variations. The remaining parameters of the described computational process were set equal to $n = 30$, $\Delta x = 1/30$, and $\varepsilon = 10^{-4}$.

The optimal modes were computed for the following values of the parameter $\alpha = 0$; $0.025 \cdot 2^n$, $n = 0, 1, \ldots, 8$. A constant thickness distribution $h_0(x) = {}^1/_2$ was selected as initial approximation h_0 of the function $h(x)$ for each of the mentioned α values, and the deflection distribution and value of the frequency squared Ω^2, corresponding to a constant thickness distribution, were determined in the first stage of the computations.

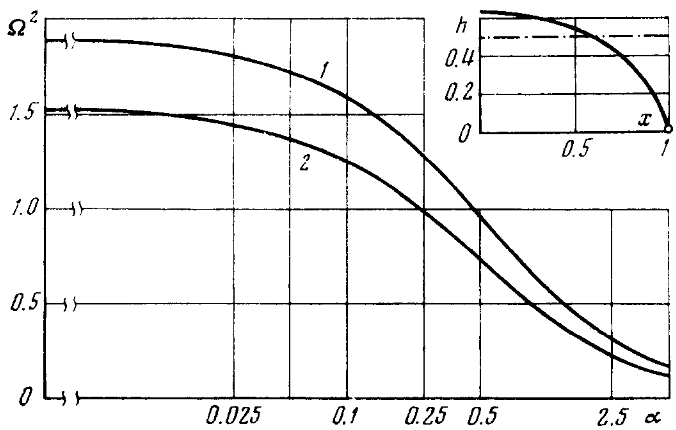

Fig. 1

Shown in Fig. 1 are dependences of the frequency squared Ω^2 on α for optimal plates (curve *1*) and for constant-thickness plates (curve *2*). Also shown in the optimal thickness distribution (symmetric relative to $x = 0$) for $\alpha = 6.4$.

The function $h(x)$ reaches a maximum for $x = 0$ and tends to zero as x tends to ± 1, i.e. upon approaching the hinge-supported plate ends. The dash-dot line shows the initial approximation for the function $h_0(x)$. Let us note that the gain in Ω^2 is 23.6% for $\alpha = 0$ and 38% for $\alpha = 1.6$.

5. Questions of the foundation. Let us consider the eigenvlaue boundary value problem (1.1),(3.1) for the linear integro-differential equation (3.1) under the boundary conditions (1.11), to whose solution reduces an investigation to harmonic vibrations of an elastic plate in an ideal fluid. Let us show that the eigennumbers of the problem (1.11),(3.1) are real, and the Rayleigh variational principle ([15]) can be used to seek the minimal (first) eigenvalue. To prove that the eigennumbers are real and the Rayleigh principle applicable, as follows from general theorems (see [15], for example), it is sufficient to establish that the boundary value problem under consideration is self-adjoint and completely definite.

The proof of self-adjointness of the boundary value problem (1.11),(3.11) reduces to verifying the equality

$$(V,\ L_iU) = (U,\ L_iV) \tag{5.1}$$

where U and V are continuously quadruply differentiable functions satisfying the boundary conditions (1.11) and L_1 and L_2 denote the following linear operators ($L = L_1 - \Omega^2 L_2$):

$$L_1U = \frac{d^2}{dx^2}\left(h^3 \frac{d^2U}{dx^2}\right), \quad L_2 = hU + \alpha \int_{-1}^{1} K(t, x)\, U(t)\, dt \tag{5.2}$$

The parentheses in (5.1) denote the scalar product, understood to be an integral of the product of the functions in the parentheses over the segment $[-1, 1]$. Self-adjointness of the operator L_1 is established by integrating by parts and using conditions (1.11)

$$(V,\ L_1U) = \int_{-1}^{1} V \frac{d^2}{dx^2}\left(h^3 \frac{d^2U}{dx^2}\right) dx = \int_{-1}^{1} U \frac{d^2}{dx^2}\left(h^3 \frac{d^2V}{dx^2}\right) dx = (U,\ L_1V) \tag{5.3}$$

To verify self-adjointness of the second operator in (5.2), we first establish the symmetry of the kernel $K(t, x)$. Taking into account that $r(t, x) = 1/r(x, t)$ (see (2.8)), and performing elementary calculations, we obtain

$$K(t, x) = \frac{1}{\pi} \ln \left| \frac{1 + r(t, x)}{1 - r(t, x)} \right| = \frac{1}{\pi} \ln \left| \frac{r(x, t) + 1}{r(x, t) - 1} \right| = \frac{1}{\pi} \ln \left| \frac{1 + r(x, t)}{1 - r(x, t)} \right| = K(x, t) \tag{5.4}$$

Furthermore, using the symmetry property remarked in (5.4), let us establish the equality required

$$(V,\ L_2U) = \int_{-1}^{1} hVU\, dx + \alpha \int_{-1}^{1} V(x)\, dx \int_{-1}^{1} U(t) K(t, x)\, dt = \int_{-1}^{1} hUV\, dx + \alpha \int_{-1}^{1} U(x)\, dx \int_{-1}^{1} V(t) K(t, x)\, dt = (U,\ L_2V) \tag{5.5}$$

We conclude on the basis of the equalities (5.3) and (5.5) proved, that the boundary value problem (1.11),(3.1) is self-adjoint.

Let us prove that the problem (1.11),(3.1) is completely definite, i.e. that

$$(U,\ L_iU) > 0, \qquad (i = 1,2) \tag{5.6}$$

for all quadruply continuously differentiable functions $U(x)$ which are not identically zero and satisfy the boundary conditions (1.11).

The inequality (5.6) is established directly in the case $i = 1$

$$(U, L_1U) = \int_{-1}^{1} U \frac{d^2}{dx^2}\left(h^3 \frac{d^2U}{dx^2}\right) dx = \int_{-1}^{1} h^3 \left(\frac{d^2U}{dx^2}\right)^2 dx > 0 \qquad (5.7)$$

Let us prove that the operator L_2 is positive. Let us write down the corresponding scalar product

$$(U, L_2U) = \int_{-1}^{1} hU^2\, dx + \alpha \int_{-1}^{1} U(x)\, dx \int_{-1}^{1} K(t, x) U(t)\, dt \qquad (5.8)$$

and let us convert the second integral in the right side of (5.8) by using the relationsships (1.13), (2.7) and (2.9)

$$\alpha \int_{-1}^{1} U(x)\, dx \int_{-1}^{1} K(t, x) U(t)\, dt = -\alpha \int_{-1}^{1} (\Phi^+ - \Phi^-) U\, dx = \qquad (5.9)$$

$$-\alpha \left(\int_{-1}^{1} \Phi^+ \left(\frac{\partial \Phi}{\partial y}\right)^+ dx - \int_{-1}^{1} \Phi^- \left(\frac{\partial \Phi}{\partial y}\right)^- dx \right) = -\alpha \oint_S \Phi \frac{\partial \Phi}{dn} dS$$

Here n denotes the external normal to the plate surface. The contour integral in (5.9) is taken over the plate surface S, i.e. over both edges of the slit $y = 0,\ -1 \leqslant x \leqslant 1$.

Applying Green's formula to the contour integral, we obtain

$$-\alpha \oint_S \Phi \frac{\partial \Phi}{\partial n} dS = \alpha \int_V (\nabla \Phi)^2\, dV \qquad (5.10)$$

To clarify the manipulations (5.9) and (5.10) we made, let us recall that by construction (see Sect. 2), the quantity

$$\int_{-1}^{1} U(t) K(t, x)\, dt$$

defined for an arbitrary comparison function U is the difference $\Phi^+ - \Phi^-$ between the boundary values of the harmonic function Φ $(\Delta \Phi = 0)$. The required estimate

$$(U, L_2U) = \alpha \int_V (\nabla \Phi)^2\, dV + \int_{-1}^{1} hU^2\, dx > 0 \qquad (5.11)$$

results from (5.8) – (5.10).

Hence, V in (5.10), (5.11) is understood to be the domain occupied by the fluid, i.e. the exterior of the slit $y = 0,\ -1 \leqslant x \leqslant 1$.

From the relationships (5.3), (5.5), (5.7) and (5.11) established, the reality of the eigenvalues of the problem (1.11), (3.1) and the applicability of the Rayleigh principle follow.

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ELASTIC EQUILIBRIUM OF ANISOTROPIC SHELLS REINFORCED BY STIFFENER RIBS

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A fundamental solution of the theory of shallow anisotropic shells is constructed, the principal part is extracted and some of its properties are studied. A procedure is indicated for constructing the Green's function for a finite shell. The solution constructed is used in investigating the state of stress of an anisotropic shell in the neighborhood of the point of application of a concentrated force. A solution is given for the problem of elastic equilibrium of an anisotropic shell rein-